Universal Weak Lensing Distortion of Cosmological Correlation Functions

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Gravitational lensing affects observed cosmological correlation functions because observed images do not coincide with true source locations. We treat this universal effect in a general way here, deriving a single formula that can be used to determine to what extent this effect distorts any correlation function. We then apply the general formula to the correlation functions of galaxies, of the 21-cm radiation field, and of the CMB.

I. INTRODUCTION

Gravitational lensing has emerged as a powerful cosmological tool. The spectacular images provided by strong lensing [1, 2, 3] together with the robust parameter determination anticipated with weak lensing [4, 5, 6, 7, 8, 9, 10, 11] are just two of the exciting developments in the field. When astronomers look for lensing, they have demonstrated that they can find it. Lensing can also contaminate observations of correlation functions because the images of objects are displaced from the true source positions. For example, it has long been known that lensing smoothes out the spectrum of the cosmic microwave background (CMB) [12, 13, 14]. The effect of lensing on galaxy-galaxy correlations has been re-examined by several groups recently [15, 16, 17, 18, 19, 20, 21]. And a number of groups are anticipating the importance of lensing on future observations of neutral hydrogen via long wavelength measurements of the redshifted 21 cm line [22, 23, 24].

In this work we present a unified treatment of the lensing contamination of correlation functions. First, we show that weak lensing contaminates *any* cosmological correlation function. Whereas previous studies have quantified its effect on some specific correlation functions (for instance, CMB temperature and polarization [13] and baryon oscillations through the galaxy correlation function [17]), we focus here on the *generality* of the result. This general framework allows us to understand when and for what correlation functions weak lensing represents a sizeable or a negligible effect. By working in real space, we also hope to provide physical intuition for some of the properties of the lensing effects, which is harder to obtain from a pure k-space calculation. Second, we consider the dependence of the lensing effect on the orientation of the source separation with respect to the line of sight. We also include the lensing-induced time delay as the longitudinal companion effect to the transverse lensing deflections. Finally, we show how the general framework presented here can be applied to different correlation functions: we retrieve previous results for the CMB, supplement and correct previous galaxy-galaxy results [17], and apply the formalism to the anisotropy induced by lensing of 21-cm radiation.

The paper is organized as follows. In Sec. II a general formalism to calculate the weak lensing contribution to cosmological correlations is introduced along with a more physical understanding of the different quantities involved. In Sec. III the general formalism developed is applied to three different cases: correlation of galaxies, of the cosmic microwave background and of the 21-cm radiation. Finally, Sec. IV contains some discussion together with concluding remarks. The calculations are relegated to the appendices.

II. GENERAL FORMALISM

Weak lensing, i.e. the effect of small potential differences in the intervening space on the path of light, consists to first order of two effects: a *transverse deflection* of the photons from a straight path, and a *time delay* (Shapiro delay) along the path. Together, these effects result in a three-dimensional displacement of the apparent source position.

Consider two physical observables $A(\vec{x}_a)$ and $B(\vec{x}_b)$

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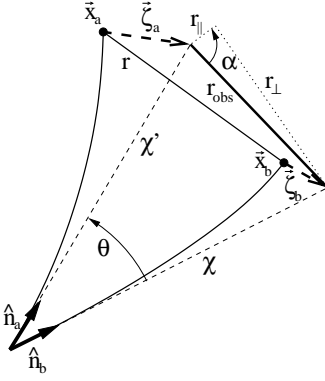[‡]Electronic address: vallinot@iap.fr

FIG. 1: Gravitational lensing bends photon trajectories. The sources, physically located at $\vec{x}_a$ and $\vec{x}_b$ are observed at $\vec{x}_a+\vec{\zeta}_a$ and $\vec{x}_b+\vec{\zeta}_b$ (in direction $\hat{n}_a$ and $\hat{n}_b$). α denotes the angle with the line of sight of the observed separation vector, $\vec{r}_{obs}$.

and denote the *observed* values of A and B as $\tilde{A}(\vec{x}'_a)$ and $\tilde{B}(\vec{x}'_b)$. In this section we keep the nature of these cosmological observables *completely unspecified*. The form of the results depends neither on the actual physical observables that are being correlated nor on the nature of the source of the signal that is used to measure them. Since the geodesics light travels on are perturbed by the intervening distribution of matter, the measured values of the observables refer to physical points that are displaced with respect to the observed ones.

Consider the arrangement depicted in Fig. 1. Since gravitational lensing perturbs the photons' trajectories, the two sources – located at $\vec{x}_a$ and $\vec{x}_b$ – are observed at $\vec{x}'_a = \vec{x}_a + \vec{\zeta}_a$ and $\vec{x}'_b = \vec{x}_b + \vec{\zeta}_b$. The observed distance $\vec{r}_{obs}$ between the sources will in general differ from the true distance $\vec{r}$. Whereas isotropy demands that the unlensed correlation function depends only on the distance r, the lensed correlation function depends on the angle α that $\vec{r}$ makes with the line of sight. Note that any difference in the line-of-sight angle of $\vec{r}$ and $\vec{r}_{obs}$ is a higher order correction, and we hence consider them to be equal for the perturbative approach adopted in this paper. Whenever the correlation function of two cosmological observables is measured, such measurement is subject to a modification arising because of the displacement of the apparent source positions $\vec{x}'_a$, $\vec{x}'_b$:

$$\langle \tilde{A}(\vec{x}'_a)\tilde{B}(\vec{x}'_b)\rangle = \langle A(\vec{x}'_a - \vec{\zeta}_a)B(\vec{x}'_b - \vec{\zeta}_b)\rangle. \quad (1)$$

By Fourier transforming Eq. (1) and introducing the power spectrum of the two observables

$$\langle A(\vec{k}_a,\eta_a)B^*(\vec{k}_b,\eta_b)\rangle \equiv (2\pi)^3\delta(\vec{k}_a - \vec{k}_b)P_{AB}(k_a,\eta_a,\eta_b), \quad (2)$$

it is straightforward to show that

$$\langle \tilde{A}(\vec{x}_a)\tilde{B}(\vec{x}_b)\rangle = \int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}_a-\vec{x}_b)}P_{AB}(k,\eta_a,\eta_b)$$
$$\times \langle e^{i\vec{k}\cdot(\vec{\zeta}_a-\vec{\zeta}_b)}\rangle, \quad (3)$$

assuming that the observables A, B are uncorrelated with the lensing deflection field $\vec{\zeta}$. Here η_A and η_B denote the conformal times at which light was emitted at the sources to reach us today.

Equation (3) allows us to express the effect that weak lensing has on the unlensed correlation function. In other words, it quantifies the effect that the matter distribution – responsible for lensing – has on the observed distribution of the physical observables. Note that the weak lensing modification depends only on the difference $\vec{\zeta}_a - \vec{\zeta}_b$ of the deflections in both observables, not on the absolute magnitude of each. Assuming a flat Universe, as we do throughout, these distortions are given by the following integrals along the line of sight:

$$\zeta^j_{a,\perp} = 2\int_0^{\chi_a} d\chi(\chi_a - \chi)\nabla^j\Phi\left[\chi\vec{\theta}_a,\chi\right] \quad (4)$$

$$\zeta_{a,\parallel} = -2\int_0^{\chi_a} d\chi\, \Phi\left[\chi\vec{\theta}_a,\chi\right]. \quad (5)$$

Here the subscripts $\perp$ and $\parallel$ indicate directions transverse and parallel to the line of sight, respectively. In the transverse case, there are two such direction, indexed here with $j = 1, 2$. The subscript a refers to the position of the observable A, fully specified by the direction $\vec{\theta}_a$ and the comoving distance χ_a. In the integral over χ, the potential is to be evaluated along the line of sight so its argument is $\vec{x}(\chi) = [\chi\vec{\theta}_a,\chi]$ valid in the small angle limit we consider here. There is one important difference between the parallel distortion $\zeta_\parallel$ and the transverse distortion $\zeta_\perp$: the latter has an extra factor of $(\chi_a - \chi)\nabla$. The potential typically varies on small scales, so this factor is of order $\chi/r \gg 1$. The transverse distortion therefore produces the dominant change in correlation functions.

By neglecting all non-Gaussianity present in the lensing field, it is possible to evaluate Eq. (3) to all orders, thus generalizing the results obtained by Challinor and Lewis [13] in the case of CMB lensing. However, it is not clear to what extent this improves the accuracy, since non-Gaussian terms, induced by gravity during structure formation, appear in the expansion at all orders above the second and may play a significant role especially at low redshift and small scales. This issue can be resolved by tracing light rays through N-body simulations of cosmological volumes (e.g., [25]). In the case of CMB lensing, there are indications that deviations from the predictions of the calculation of [13] which neglect non-Gaussianities in the deflection field begin to appear at $\ell \gtrsim 2000$ in the deflection field. A detailed analysis of such a general treatment and of its range of validity, requiring ray-tracing measurement of the probability density function (PDF) of the displacements $\vec{\zeta}$, will be the focus of a forthcoming work. We therefore briefly outline this general formalism in appendix B and for the rest of this work we will follow the perturbative approach.

In order to proceed, we expand the exponential in Eq. (3) to second order in the deflection. Note that all odd terms in this expansion vanish due to symme-

try, so that the expansion is good to fourth order. This is sufficient to quantify the weak lensing impact on the correlation function for sufficiently large separations. Expanding Eq. (3) above we then have

$$\langle \tilde{A}\tilde{B} \rangle \approx \langle AB \rangle + \frac{Z^{ij}}{r^2}$$
$$\times \left[r_i r_j \frac{d^2 \langle AB \rangle}{dr^2} + r \frac{d\langle AB \rangle}{dr} \left(\delta_{ij} - \frac{r_i r_j}{r^2} \right) \right], \quad (6)$$

where $\vec{r} = \vec{x}_a - \vec{x}_b$ denotes the observed separation between the sources and the 3×3 distortion correlation tensor is defined as

$$Z^{ij} \equiv \frac{\langle \zeta_a^i \zeta_a^j \rangle + \langle \zeta_b^i \zeta_b^j \rangle}{2} - \langle \zeta_a^i \zeta_a^j \rangle$$
$$\equiv \begin{pmatrix} T - D/2 & 0 & -V_a \\ 0 & T + D/2 & 0 \\ -V_b & 0 & S \end{pmatrix}. \quad (7)$$

The second definition here of the matrix elements holds if we choose $\vec{x}_a$ to lie along the $\hat{z}$ axis and both positions to lie in the $x - z$ plane, so that $\vec{x}_a$ and $\vec{x}_b$ have coordinates $\vec{x}_a = (0, 0, \chi_a)$ and $\vec{x}_b = (r_\perp, 0, \chi_a + r_\parallel)$ respectively. Note that due to the azimuthal symmetry of the lensing displacements around the line of sight, we can always choose such coordinates without loss of generality. In appendix A we derive expressions for the elements of the distortion tensor using the Limber approximation. The correlation of the distortions transverse to the line of sight decompose into a piece which is similar along both x and y directions

$$T(\chi, r_\parallel, r_\perp) = T_1(\chi, r_\parallel) + T_2(\chi, r_\perp) \quad (8)$$

with

$$T_1(\chi, r_\parallel) \equiv r_\parallel^2 \int_0^\chi d\chi' \int_0^\infty \frac{dk k^3 P_\Phi(k, \chi')}{2\pi} \quad (9)$$

$$T_2(\chi, r_\perp) \equiv \int_0^\chi d\chi' \int_0^\infty \frac{dk k^3 P_\Phi(k, \chi')}{\pi} (\chi - \chi')^2$$
$$\times (1 - J_0(k r_\perp \chi'/\chi)) \quad (10)$$

and one which differs in the two transverse directions:

$$D(\chi, r_\perp) = 2 \int_0^\chi d\chi' (\chi - \chi')^2$$
$$\times \int_0^\infty \frac{dk k^3 P_\Phi(k, \chi')}{\pi} J_2(k r_\perp \chi'/\chi). \quad (11)$$

Note that both T_2 and D, which depend only on the transverse distance between the two background objects, vanish in the limit that $r_\perp = 0$. Since we will see that these two terms dominate the distortion of the correlation function, changes to the correlation functions have a characteristic dependence on the *projected* separation $r_\perp$, peaking when $r_\perp = r$. The variance of the displacements along the line of sight due to time delay is:

$$S(\chi, r_\perp) = 2 \int_0^\chi d\chi' \int_0^\infty \frac{dk k P(k, \chi')}{\pi}$$
$$\times [1 - J_0(k r_\perp \chi'/\chi)]. \quad (12)$$

Finally, the displacement along the line of sight is slightly correlated with the transverse distortion; the relevant combination is

$$V_a + V_b = 2r_\parallel \int_0^\chi d\chi' \int_0^\infty \frac{dk k^2 P(k, \chi')}{\pi} J_1(k r_\perp \chi'/\chi). \quad (13)$$

It is useful to estimate the order of magnitude of the corrections. First consider Eqs. (4, 5). The RMS of the perpendicular distortion is of order $\Phi_{\rm RMS} \chi^2 / r$, where χ is a typical cosmological distance, of order a Gpc, and r is a typical distance over which the potential varies, typically much smaller. The RMS of the parallel distortion is seen to be smaller by a factor of (r/χ). These estimates translate well into the respective variances as encoded in the functions T_2 and D. Both T_2 and D are seen from Eqs. (10, 11) to be of order $k^3 P_\Phi(k) k\chi^3$ since the line of sight variable χ' is order χ. The variance of the gravitational potential is roughly $\Delta_\Phi^2(k) \sim k^3 P_\Phi(k)$, and the k integral picks out values of $k \sim r^{-1}$. So both T_2 and D are of order $\Delta_\Phi^2(1/r)(\chi^2/r)^2 \times (r/\chi)$, where $\Delta_\Phi^2(k)$ is equal to $\Phi_{\rm RMS}^2 \sim 10^{-9}$ on large scales ($k \lesssim 0.01 h/{\rm Mpc}$) and suppressed on smaller scales due to the transfer function. The extra factor of r/χ is the standard Limber suppression due to cancellation along the line of sight (only modes with k_z small contribute to the variance). On large scales, the relevant dimensionless quantities T_2/r^2, D/r^2 thus are of order $\Phi_{\rm RMS}^2 \chi^3/r^3 \sim 10^{-6} (100 \, {\rm Mpc}/h/r)^3$. They increase towards small scales, however not as quickly as r^{-3}, since the variance of the potential is suppressed by the transfer function towards smaller scales. In other words, due to the suppressed power on small scales, photons from both directions are deflected more and more coherently, and r_{obs} is very close to r which is reflected by reduced values of T_2 and D.

The first term in the transverse deflection correlation, T_1 as defined in Eq. (9), represents the difference between the lensing experienced by A and B due to their different distances from us, so this part of the transverse lensing is suppressed by a factor of $r_\parallel^2/\chi^2$ and can usually be neglected. Similar estimates show that S and $V_a + V_b$ are both smaller than the transverse variances T_2 and D by a factor of $(r/\chi)^2$ as expected from a cursory examination of Eqs. (4,5). So we expect, and numerical work confirms this, that T_2 and D dominate the corrections to correlation functions, and the corrections will be of order $\Delta_\Phi^2(1/r)(\chi/r)^3$.

Therefore, corrections to cosmological correlations due to weak lensing are small and given by:

$$\langle \tilde{A}\tilde{B} \rangle(r, r_\perp, \chi) \approx \langle AB \rangle(r, \chi) + T_2(\chi, r_\perp) \frac{2}{r} \frac{d\langle AB \rangle}{dr}$$
$$+ \left(T_2(\chi, r_\perp) - \frac{D(\chi, r_\perp)}{2} \right) \frac{r_\perp^2}{r^2} r \frac{d}{dr} \left[\frac{1}{r} \frac{d\langle AB \rangle}{dr} \right]. \quad (14)$$

Two general features of this equation are worth pointing out: first, if the correlation function is close to a
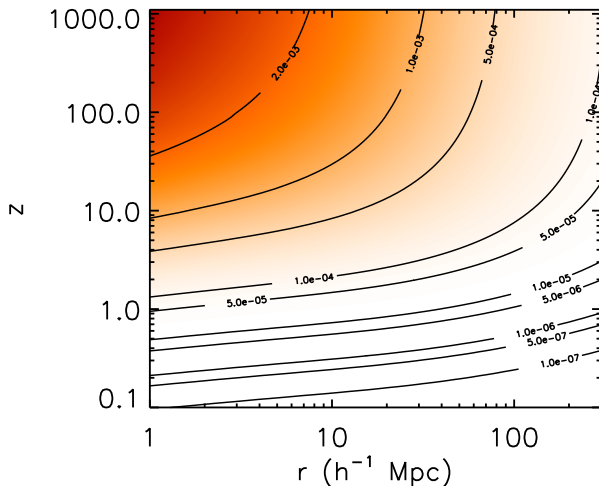
FIG. 2: Contour plot of T_2/r^2 [Eq. (10)] determining the relative transverse displacement as a function of the source separation r and source redshift z (assumed equal for both sources, so $r_\perp = r$).

FIG. 3: Same as Fig. 2 but for D/r^2, where D is defined in Eq. (11).

pure power-law, both derivative terms will be of order $\langle AB \rangle / r^2$. Hence, the relative lensing-induced effect on the correlation function will be given by T_2/r^2 and D/r^2, which are shown in Fig. 2 and 3. Second, in case of an oscillating correlation function, the lensing contribution can be amplified by large values of $d\langle AB \rangle/dr$ and $d^2 \langle AB \rangle/dr^2$. Further, lensing will tend to smooth out the oscillations: at a local minimum of $\langle AB \rangle$, the first derivative vanishes, while the second derivative is positive, so that the observed correlation is increased by lensing. At a local maximum, the opposite holds. This feature is already well-known in ℓ-space in case of the CMB.

Equation (14) holds for any cosmological correlation function. It applies equally well to point-like sources and to diffuse backgrounds, to galaxies and QSO, to CMB and to the 21-cm radiation (albeit in the case of discrete sources magnification bias effects might provide the dominant distortion of the correlation function). Given a particular matter distribution, it is sufficient to evaluate the functions T_2 and D *once* to be able to calculate the effect that weak lensing has on the correlation function of *any* cosmological observable.

The functions T_2/r^2 and D/r^2 are shown for a concordance ΛCDM cosmology with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$ (which will be assumed throughout) in Figs. 2 and 3. We used the non-linear matter power spectrum from `halofit` [26] in the calculation. As the positions A and B move out to higher redshift, the lensing effects become larger since the longer path lengths lead to larger RMS deflections. At fixed redshift, both T_2 and D become larger as r gets bigger since the displacements of the path from the two points to us cease to be correlated and hence experience independent deflections. However, as discussed
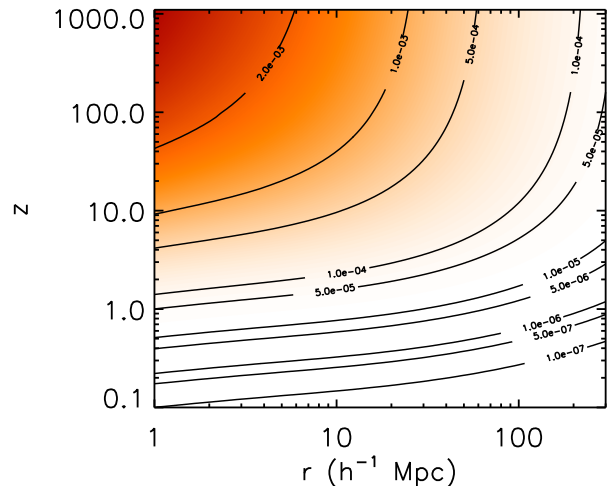
above, the relative effect of lensing distortions will generically be of order T_2/r^2, D/r^2. These quantities decrease as r increases, as illustrated in Figs. 2 and 3. Hence, in the absence of features in the unlensed correlation function considered, the effect of lensing is likely to be most important on small scales. Since the perturbative expansion treatment we are using is valid under the condition that $(T_2/r^2, D/r^2) \ll 1$, Figs. 2 and 3 illustrate that the approximation used here is always applicable: it will yield a good approximation to the size of the lensing effect. If the effect is only marginally observable, this approximation should be sufficient. If however the desired precision on lensing effect itself is high, as, e.g. in the case of CMB lensing in order to improve cosmological parameter constraints, higher-order corrections will have to be taken into account.

The value of the correlator of the longitudinal displacements $S/r^2 = Z^{33}/r^2$ is shown in Fig. 4. The longitudinal displacement effect (time delay) is clearly much smaller than the perpendicular one, as found earlier in [27].

III. APPLICATIONS

We are now in a position to apply the above results to the correlation function of three different kinds of sources: galaxies, the CMB, and the 21-cm radiation background. The CMB is an angular measurement so all photons come from the same comoving distance meaning that $r_\parallel = 0$ and the dependence is solely on $r = r_\perp$. For galaxies and 21cm, the orientation of the radius vector with respect to the line of sight is a free parameter, so the smoothing depends on $r_\perp$ even when r is fixed.
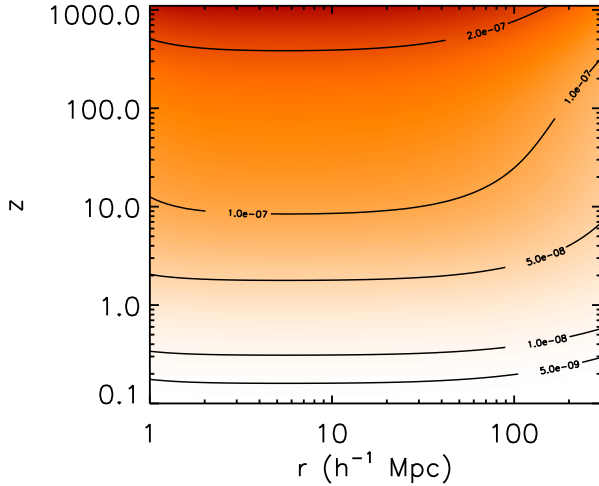
FIG. 4: Contour plot of S/r^2 [Eq. (12)] determining the relative longitudinal displacement as a function of the source separation r and source redshift z (assumed equal for both sources).
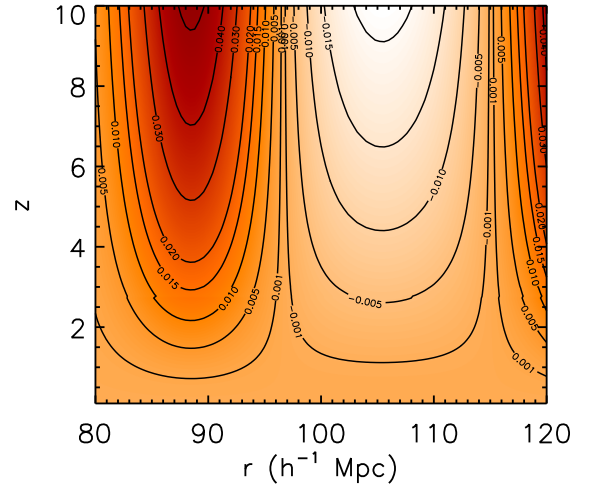

FIG. 5: Relative deviation from the unlensed correlation function of the galaxy correlation function including effects of lensing deflections, near the baryon acoustic scale. Lensing smoothes the BAO feature in the galaxy correlation function at the percent level.

A. Galaxies

The effect of weak lensing on the galaxy correlation function has been taken into account in previous work in the context of the analysis of the impact that lensing has on the determination of the sound horizon of baryon oscillations [17]. The ratio of the lensing induced term to the unlensed correlation function is plotted in Fig. 5. Lensing smoothes the BAO bump at the percent level. Note that since the lensing effect multiplies derivatives of the unlensed correlation function, the relative effect of lensing will be independent of any galaxy bias, in contrast to the magnification bias, whose relative effect is $\propto 1/b$ or $\propto 1/b^2$, depending on redshift [16]. We point out that Eqs. (6-8) of [17] are recovered using Eq. (14) above once the dependence of the kernels on the angle θ is correctly taken into account in [17].

Fig. 6 shows the angular dependence of the correlation function for galaxies at redshift 3. The characteristic angular dependence illustrated in Fig. 6 may make this effect detectable. Note that due to the smallness of the time-delay effect (S/r^2, Fig. 4), the lensing contribution essentially vanishes for $\alpha \to 0$.

B. Cosmic Microwave Background

The smoothing effect of lensing on the CMB power spectrum was computed long ago [12] in multipole space. The formalism established here allows for a simple calculation of this same effect in angular space.

The angular temperature correlation function of the CMB is given by:

$$w_{TT}(\theta) = \sum_{l=2}^{\infty} \frac{2l+1}{4\pi} C^{TT}(l) \, P_l(\cos\theta), \qquad (15)$$

where P_l denote the Legendre polynomials. Applying the approach presented here, we can calculate the lensing effect on the angular correlation function by evaluating Eq. (14). Then, $\langle AB \rangle(r)$ stands for the temperature correlation function $\xi_{TT}(r) = w_{TT}(r/\chi_*)$, where χ_* is the distance to the last scattering surface. For this, we need the first and second derivatives of $\xi_{TT}(r)$ which are given by:

$$\frac{d}{dr}\xi_{TT}(r) = \frac{-\sin\theta}{\chi_*} \frac{dw_{TT}}{d\cos\theta}$$
$$\frac{d^2}{dr^2}\xi_{TT}(r) = \frac{-\cos\theta}{\chi_*^2} \frac{dw_{TT}}{d\cos\theta} + \frac{\sin^2\theta}{\chi_*^2} \frac{d^2 w_{TT}}{d\cos\theta^2} \quad (16)$$

The derivatives of w_{TT} with respect to $\cos\theta$ can be carried out using the Legendre polynomial relation:

$$P_l'(x) = \frac{l}{x^2-1}(xP_l(x) - P_{l-1}(x)) \qquad (17)$$

Fig. 7 shows the lensed and unlensed CMB angular correlation functions in the region of the baryon acoustic feature. One can discern a slight smoothing effect of lensing in real space, as pointed out after Eq. (14). Fig. 8 shows the difference between the lensed and unlensed correlation functions calculated in this approach (thick line). The unlensed $C^{TT}(l)$ were obtained from CAMB [28]. In the figure we also show the correlation function
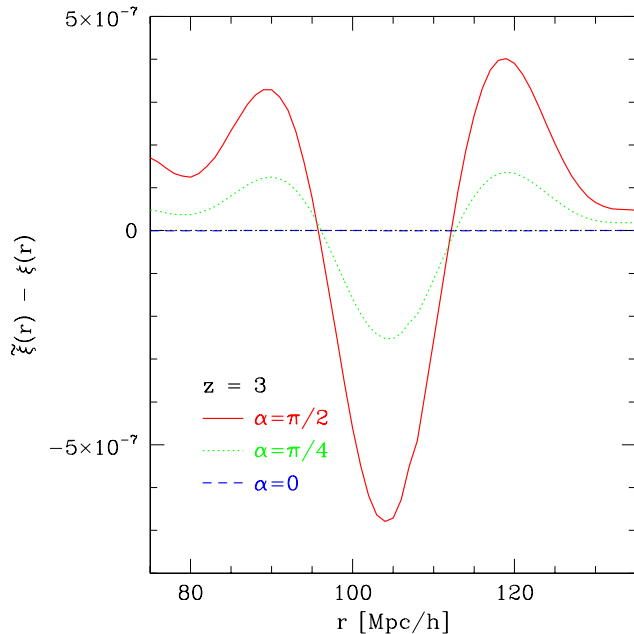
FIG. 6: Difference between the lensed and unlensed matter correlation function at $z = 3$ near the BAO feature. The three lines shown are for different orientations of the radius vector with respect to the line of sight: perpendicular (solid/red), at 45° (dotted/green), and parallel to the line of sight (dashed/blue). The lensing contribution shows a strong dependence on the orientation of the separation vector.

obtained from the lensed $C^{TT}(l)$ given by CAMB using Eq. (15) (thin line). Clearly, the calculations in the two different approaches agree for $\theta \gtrsim 0.1°$ corresponding to $r \gtrsim 15\mathrm{Mpc}/h$. Note that results from N-body simulations [25] show deviations from the results of the CAMB code for $\ell \gtrsim 2000$, roughly corresponding to $\theta \lesssim 0.03°$; however, it is not straightforward to convert the scale where deviations appear in multipole space to a corresponding angular scale in real space.

C. 21-cm Surveys

In principle, redshifted 21 cm radiation encodes information about the 3D distribution of neutral hydrogen in the universe. This distribution is sensitive to both inhomogeneities in the matter and in the free electron density. It is not our purpose here to compute the complicated correlation function that results. Rather, we note that the real space calculation presented here is particularly simple to apply to any model of reionization (see Ref. [22] for a careful discussion of the complications that arise in Fourier space).

For the purposes of this paper, we use the 21cm predictions for the "dark ages" before reionization calculated in [29]. We use the 21cm spin temperature correlation coefficients $C^{TT}(\ell)$ at $z = 50$, in the same way as outlined
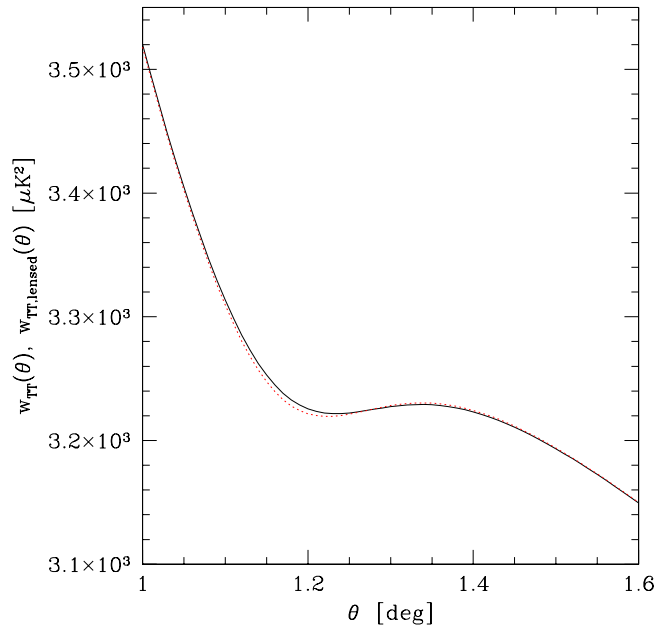

FIG. 7: Lensed (solid) and unlensed (dotted) CMB angular correlation functions. A slight smoothing effect washing out the baryon acoustic feature can be seen.

above in the case of the CMB. Fig. 9 shows the relative lensing effect on the 21cm angular correlation function. It is rather small compared to that of the CMB due to the overall smoothness of the 21cm angular correlation function in the dark ages. Note that at lower redshifts, $z \lesssim 12$, we expect the 21cm correlation function to show a stronger baryonic signature [30], and hence expect a higher lensing effect possibly of importance to cosmological parameter constraints [31, 32, 33, 34].

IV. DISCUSSION

Gravitational lensing affects cosmological correlation functions in two ways. First, since the geodesics which photons travel on are sensitive to the distribution of matter between source and observer and since the latter is locally inhomogeneous thanks to structure formation, weak lensing acts by displacing the sources' observed (as opposed to true) positions. The correlation function that is measured from observed data therefore also includes the contribution from these lensing deflections. Second, weak lensing also acts on observations by focusing or defocusing the geodesics' congruences, thus altering the observed brightness of a source. This latter effect, known as *magnification bias*, is generally larger than the former [16, 21] but it applies only to pointlike sources (see e.g. [16, 17, 20] for studies of the effect of magnification bias on two- and three-point correlation functions). Both these effects are small but potentially relevant in the present age of precision cosmology. In this work we
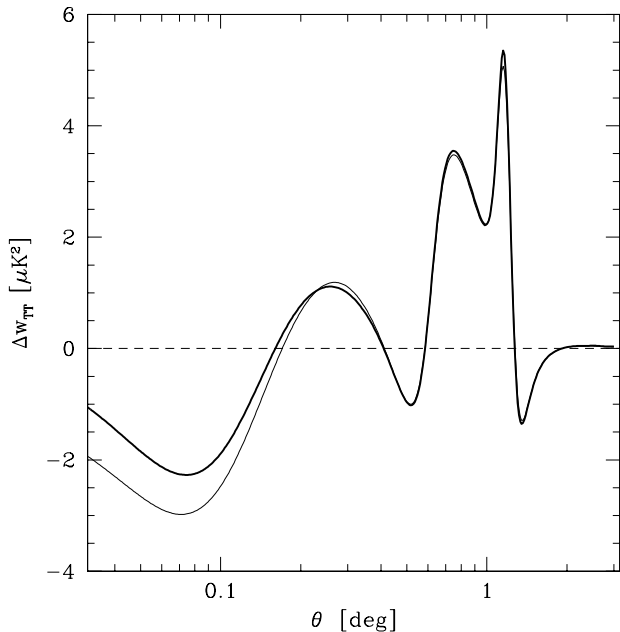
FIG. 8: Difference between the lensed and unlensed CMB angular correlation function $w_{TT}(\theta)$. The thick line shows the calculation using the first-order approach presented here, while the thin line is the lensed w_{TT} obtained from CAMB.
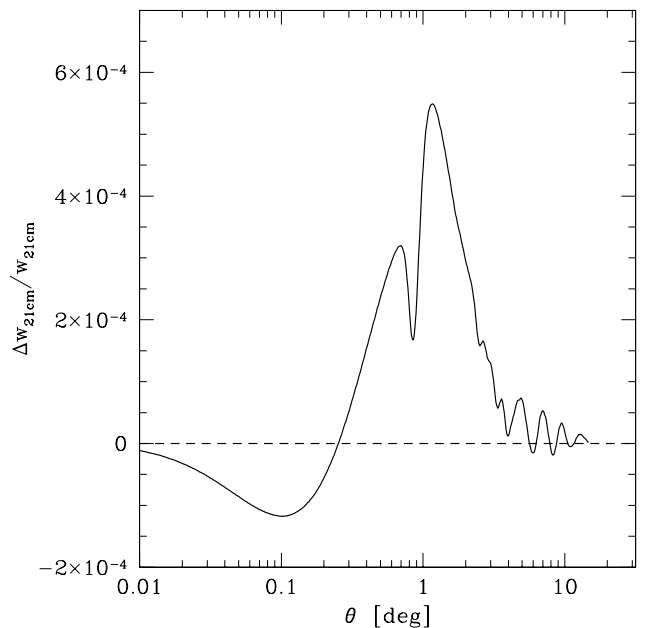

FIG. 9: Relative lensing effect on the angular correlation function of 21cm emission from the dark ages. We use the cross-correlation coefficients from [29]. The redshift is $z = 50$, and the bandwidth chosen is 0.01 MHz.

focused on the first of these effects and we derived a *general* perturbative formula that can be used to quantify it.

The results of the present work depend only on the following two assumptions: that photons travel on geodesics and that higher order corrections to the correlation function (arising from higher order correlators of the lensing induced displacements) can safely be neglected. The first of this assumption has two consequences of opposite nature. The positive consequence is that the result obtained in this work is general and applies to *any* cosmological correlation function, regardless of the nature of the source and of the physical observables that are being measured and correlated. Moreover, the correction terms that appear in Eq. (14) and that quantify the effect of lensing depend only on the power spectrum of matter: given a specific cosmological model they need to be evaluated only once and they can then be applied to any correlation function. As such, they also represent a map that will tell whether weak lensing will have a relevant or an irrelevant role in the measurement of a given correlation function *before* the actual measurement is carried out. The negative consequence, on the other hand, is that this lensing-induced distortions represent a theoretical systematic that will *always* be present regardless of the precision of the instruments used to carry out the observation. In other words, even if "perfect observations" free of any systematics could be carried out, this lensing-induced noise would still creep into the correlation function that would be calculated using those

data. This lensing-induced modification of the correlation may be avoided only by reconstructing a map of the lensing potential (e.g., [35]). Another avenue to a possible disentanglement of this lensing distortion from the observed correlation function – at least for sources that are not confined to a fixed redshift – could be to exploit the angular dependence of the effect as seen in Fig. 6. Furthermore, since the lensing modification depends on the derivatives of the correlation function, correlation functions that are rapidly oscillating will in general be most affected by lensing, i.e. the features will be somewhat smoothed. Conversely, in quantities that are intrinsically uncorrelated, no correlation will be induced by lensing deflections.

The second assumption entering the derivation of Eq. (14) is that terms of third and higher order are discarded when the exponential of Eq. (3) is expanded. As shown in Appendix A, this assumption corresponds to Taylor expanding the coordinate dependence of the physical observables and to retain only terms up to second order, which in turn corresponds to neglecting all contributions arising from the non-Gaussianity of the displacements' PDF. Despite the fact that under the same assumptions the general formalism outlined in Appendix B can be used to calculate the lensing distortion arising from the sum of *all* the terms appearing in the exponential of Eq. (3), it is however unclear to what extent this might really represent an improvement. If on one hand the sum of all the terms appearing in the exponential may give important contributions on small scales, it

is also true that on such small scales departures from gaussianity of the displacements' PDF – induced by the non-linear growth of structure at low redshift – could play a relevant role. The range of applicability of such a non-perturbative method and the gain in precision that it would allow on small scales are the focus of a current project.

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APPENDIX A: DERIVATION OF EQS. (6-14)

1. Perturbative Approach

Starting again from Eq. (3) it is possible to expand the exponential keeping only terms up to second order

$$
\begin{aligned}
\langle \tilde{A}\tilde{B} \rangle \approx & \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}_a - \vec{x}_b)} P_{AB}(k, \eta_a, \eta_b) \\
& \times \Big\langle 1 + i\vec{k}\cdot(\vec{\zeta}_a - \vec{\zeta}_b) \\
& - \frac{1}{2}[\vec{k}\cdot(\vec{\zeta}_a - \vec{\zeta}_b)][\vec{k}\cdot(\vec{\zeta}_a - \vec{\zeta}_b)]\Big\rangle, \quad \text{(A1)}
\end{aligned}
$$

The zero-th order (in $\vec{k}$) corresponds to the unlensed correlation function. Similarly, the first order term vanishes

because $\langle\zeta\rangle = 0$. Let's then move to consider the second term, which we'll denote as $\langle AB\rangle_2$. We can rewrite it as

$$
\begin{aligned}
\langle AB\rangle_2 &= -\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}_a-\vec{x}_b)}P_{AB}(k,\eta_a,\eta_b)\\
&\times k_ik_j\left[\langle\zeta_a^i\zeta_a^j\rangle + \langle\zeta_b^i\zeta_b^j\rangle - \langle\zeta_a^i\zeta_b^j\rangle - \langle\zeta_b^i\zeta_a^j\rangle\right]\\
&= \left[\langle\zeta_a^i\zeta_b^j\rangle - \frac{\langle\zeta_a^i\zeta_a^j\rangle + \langle\zeta_b^i\zeta_b^j\rangle}{2}\right]\\
&\times\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}_a-\vec{x}_b)}P_{AB}(k,\eta_a,\eta_b)k_ik_j \quad\text{(A2)}
\end{aligned}
$$

where we have used the fact that the displacement correlators do not depend on the integration variable k to pull them out of the integral. The above integral can be rewritten as

$$
\begin{aligned}
&\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{r}}P_{AB}(k,\eta_a,\eta_b)k_ik_j\\
&= \int\frac{d^3k}{(2\pi)^3}\frac{1}{i^2}\frac{\partial^2}{\partial r_i\partial r_j}e^{i\vec{k}\cdot\vec{r}}P_{AB}(k,\eta_a,\eta_b)\\
&= \frac{1}{i^2}\frac{\partial^2}{\partial r_i\partial r_j}\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{r}}P_{AB}(k,\eta_a,\eta_b)\\
&= -\frac{\partial^2}{\partial r_i\partial r_j}\langle AB\rangle. \quad\text{(A3)}
\end{aligned}
$$

We can then notice that

$$
\begin{aligned}
\frac{\partial^2\langle AB\rangle}{\partial r_i\partial r_j} &= \frac{d^2\langle AB\rangle}{dr^2}\frac{\partial r}{\partial r_i}\frac{\partial r}{\partial r_j} + \frac{d\langle AB\rangle}{dr}\frac{\partial^2 r}{\partial r_i\partial r_j}\\
&= \frac{r_ir_j}{r^2}\frac{d^2\langle AB\rangle}{dr^2}\\
&+ \frac{1}{r}\frac{d\langle AB\rangle}{dr}\left(\delta_{ij} - \frac{r_ir_j}{r^2}\right). \quad\text{(A4)}
\end{aligned}
$$

Putting all the pieces together we then get

$$
\begin{aligned}
\langle AB\rangle_2 &= \left[\frac{\langle\zeta_a^i\zeta_a^j\rangle + \langle\zeta_b^i\zeta_b^j\rangle}{2} - \langle\zeta_a^i\zeta_b^j\rangle\right]\\
&\times\left[\frac{r_ir_j}{r^2}\frac{d^2\langle AB\rangle}{dr^2}\frac{1}{r}\frac{d\langle AB\rangle}{dr}\left(\delta_{ij} - \frac{r_ir_j}{r^2}\right)\right] \quad\text{(A5)}
\end{aligned}
$$

which is exactly the result of Eq. (6) once we define the displacement correlator Z^{ij} as

$$
Z^{ij} \equiv \frac{\langle\zeta_a^i\zeta_a^j\rangle + \langle\zeta_b^i\zeta_b^j\rangle}{2} - \langle\zeta_a^i\zeta_b^j\rangle. \quad\text{(A6)}
$$

Finally, let's notice that the same result can be obtained in a somewhat more intuitive way simply by Taylor expanding the coordinate dependence of the observables as

$$
\tilde{A}(\vec{x}_a) = A(\vec{x}_a + \vec{\zeta}_a) \approx A_0^a + \zeta_a^iA_{,i}^a + \frac{1}{2}\zeta_a^i\zeta_a^jA_{,ij}^a, \quad\text{(A7)}
$$

where we use the shorthand notation $A_{,i}^a = \partial A/\partial x^i\big|_{\vec{x}=\vec{x}_a}$, and where the "a" index appearing on the displacement ζ_a^i and on the observables A^a specifies that these quantities refer to the physical point $\vec{x}_a$.

2. Decomposition of the Displacements' Correlator

Consider a perturbed flat FRW spacetime with metric

$$
ds^2 = a^2(\eta)\left[(-1-2\Psi)d\eta^2 + \delta_{ij}(1+2\Phi)dx^idx^j\right]. \quad\text{(A8)}
$$

The lensing induced deflection of a source at distance χ_0 is given by the following integrals along the (unperturbed) line of sight [6, 36]

$$
\begin{aligned}
\zeta_{a,\perp}^j &= 2\int_0^{\chi_0}d\chi(\chi_0-\chi)\nabla_\perp^j\Phi\\
&= 2i\int_0^{\chi_0}d\chi(\chi_0-\chi)\int\frac{d^3k}{(2\pi)^3}k_\perp^j\Phi e^{i\vec{k}\cdot\vec{x}(\chi)}, \quad\text{(A9)}\\
\zeta_{a,\parallel} &= -2\int_0^{\chi_0}d\chi\,\Phi\\
&= -2\int_0^{\chi_0}d\chi\int\frac{d^3k}{(2\pi)^3}\Phi e^{i\vec{k}\cdot\vec{x}(\chi)}. \quad\text{(A10)}
\end{aligned}
$$

With the help of the Limber approximation, it is straightforward to show that the correlators are given by

$$
\begin{aligned}
\langle\zeta_{a,\parallel}\zeta_{b,\parallel}\rangle &= 4\int_0^{\min\{\chi_a^0,\chi_b^0\}}d\chi\\
&\times\int\frac{dk}{2\pi}k\,J_0(k\chi\theta)P_\Phi(k,\chi), \quad\text{(A11)}\\
\langle\zeta_{a,\perp}^j\zeta_{b,\parallel}\rangle &= -4i\int_0^{\min\{\chi_a^0,\chi_b^0\}}d\chi\,(\chi_a^0-\chi)\\
&\times\int\frac{d^2\vec{k}_\perp}{(2\pi)^2}k_\perp^j\,e^{i\vec{k}_\perp\cdot\vec{\theta}\chi}P_\Phi[k_\perp,z(\chi)] \quad\text{(A12)}\\
\langle\zeta_{a,\perp}^i\zeta_{b,\perp}^j\rangle &= 4\int_0^{\min\{\chi_a^0,\chi_b^0\}}d\chi\,(\chi_a^0-\chi)(\chi_b^0-\chi)\\
&\times\int\frac{d^2\vec{k}_\perp}{(2\pi)^2}k_\perp^ik_\perp^j\,e^{i\vec{k}_\perp\cdot\vec{\theta}\chi}P_\Phi[k_\perp,z(\chi)], \quad\text{(A13)}
\end{aligned}
$$

where in Eqs. (A12, A13) $i,j = 1,2$ denote the two components perpendicular to the line of sight. Looking at the structure of Eqs. (A11-A13) above it is possible to notice that these quantities transform as a scalar, a vector and a symmetric tensor with respect to rotations in a plane perpendicular to the line of sight. We can therefore define right away the scalar S_{ab} as

$$
S_{ab} \equiv \langle\zeta_{a,\parallel}\zeta_{b,\parallel}\rangle, \quad\text{(A14)}
$$

and can simplify even more Eqs. (A12, A13) by extracting the components of the vector and of the tensor. In the case of the vector, for instance, it is possible to notice that the vector component is aligned along the displacement vector $\langle\zeta_{a,\perp}^j\zeta_{b,\parallel}\rangle \sim r_\perp^j$. We can then define

$$
\begin{aligned}
V_a &\equiv \frac{r_j}{r_\perp}\langle\zeta_{a,\perp}^j\zeta_{b,\parallel}\rangle \quad\text{(A15)}\\
&= 4\int_0^{\min\{\chi_a^0,\chi_b^0\}}d\chi(\chi_a-\chi)\\
&\times\int\frac{k^2\,dk}{2\pi}P_\Phi[k,z(\chi)]J_1(k\chi\theta). \quad\text{(A16)}
\end{aligned}
$$

Similarly it is possible to decompose the tensor part into its trace and its off diagonal traceless symmetric part (cfr. [13]). Letting

$$\langle \zeta_{a,\perp}^i \zeta_{b,\perp}^j \rangle = T_{ab}\delta_\perp^{ij} - D_{ab}\hat{R}_\perp^{ij}, \qquad (A17)$$

where $\hat{R}_\perp^{ij}$ is the symmetric traceless tensor defined by

$$\hat{R}_\perp^{ij} \equiv \frac{1}{r_\perp^2}\left[r_\perp^i r_\perp^j - \frac{r_\perp^2}{2}\delta^{ij}\right]. \qquad (A18)$$

and then using

$$\delta_{ij,\perp}\langle \zeta_{a,\perp}^i \zeta_{b,\perp}^j \rangle = 2T_{ab}, \qquad (A19)$$

$$r_{\perp,i} r_{\perp,j}\langle \zeta_{a,\perp}^i \zeta_{b,\perp}^j \rangle = r_\perp^2\left(T_{ab} - \frac{D_{ab}}{2}\right), \quad (A20)$$

together with the Limber approximation, it is possible to get to the following expressions for T_{ab} and D_{ab} (cfr. [13])

$$T_{ab} = 4\int_0^{\min\{\chi_a^0,\chi_b^0\}} d\chi\,(\chi_a^0 - \chi)(\chi_b^0 - \chi)$$
$$\times \int \frac{k^3 dk}{4\pi} J_0(k\chi\theta)\, P_\Phi(k,\chi), \qquad (A21)$$

$$D_{ab} = 4\int_0^{\min\{\chi_a^0,\chi_b^0\}} d\chi\,(\chi_a^0 - \chi)(\chi_b^0 - \chi)$$
$$\times \int \frac{k^3 dk}{2\pi} J_2(k\chi\theta)\, P_\Phi(k,\chi), \qquad (A22)$$

Taking the limit $\theta \to 0$ of Eqs. (A11, A16, A22) it is possible to obtain the equivalent of the above expressions for the single source case. In particular, given that $J_1(0) = J_2(0) = 0$, it is straightforward to notice that in this case $V_a = 0$ and $D_{aa} = 0$. We can then forget about the labels for D_{ab} and from here on simply identifying it with D.

Let's notice that up to this point only the Limber approximation has entered the above derivation. It is then possible to proceed further by defining

$$T \equiv \frac{1}{2}\left(T_{aa} + T_{bb} - 2T_{ab}\right). \qquad (A23)$$

Defining the following shorthand notation for the sake of brevity,

$$I_0(\chi) \equiv \int \frac{dk}{\pi} k^3 P_\Phi(k,\chi), \qquad (A24)$$

$$I_J(\chi) \equiv \int \frac{dk}{\pi} k^3 P_\Phi(k,\chi) J_0(k\chi\theta), \qquad (A25)$$

we can then rewrite Eq. (A23) above as

$$T = \frac{1}{2}\left[\int_0^{\chi_a} d\chi(\chi_a - \chi)^2 I_0(\chi) + \int_0^{\chi_b} d\chi(\chi_b - \chi)^2 I_0(\chi)\right.$$
$$\left. + \int_0^{\chi_a} d\chi(\chi_b - \chi)(\chi_a - \chi)\, I_J(\chi)\right]$$
$$= \frac{1}{2}\left[\int_0^{\chi_a} d\chi\left[(\chi_a - \chi)^2 + (\chi_b - \chi)^2\right] I_0(\chi)\right.$$
$$+ \int_{\chi_a}^{\chi_b} d\chi(\chi_b - \chi)^2 I_0(\chi)$$
$$\left. - 2\int_0^{\chi_a} d\chi(\chi_a - \chi)(\chi_b - \chi)I_J(\chi)\right]$$
$$= \frac{1}{2}\left[\int_0^{\chi_a} d\chi(\chi_b - \chi_a)^2 I_0(\chi)\right.$$
$$+ \int_{\chi_a}^{\chi_b} d\chi(\chi_b - \chi)^2 I_0(\chi)$$
$$\left. - 2\int_0^{\chi_a} d\chi(\chi_a - \chi)(\chi_b - \chi)[I_J(\chi) - I_0(\chi)]\right] \quad (A26)$$

where without loss of generality we assumed that $\chi_a < \chi_b$. Equation (A26) above is exact. We can proceed further by noting that $\chi_b - \chi_a = r_\parallel$ and that the integration limits of the second integral extend over $r_\parallel$. We then have

$$\int_0^{\chi_a} d\chi(\chi_b - \chi_a)^2 I_0(\chi) + \int_{\chi_a}^{\chi_b} d\chi(\chi_b - \chi)^2 I_0(\chi)$$
$$= r_\parallel^2\left[\int_0^{\chi_a} d\chi\, I_0(\chi) + \int_{\chi_a}^{\chi_b} d\chi\, I_0(\chi)\frac{(\chi - \chi_b)^2}{r_\parallel^2}\right]$$
$$\approx r_\parallel^2 \int_0^{\chi_a} d\chi \int \frac{dk}{\pi} k^3 P_\Phi(k,\chi) = 2T_1(\chi_a, r_\parallel), \quad (A27)$$

where to get to the last line we used the fact that while the first integral scales as $r_\parallel^2 \chi_a$, the second one scales only as $r_\parallel$ and it therefore provides a subdominant contribution. Finally, the last term of Eq. (A26) gives $T_2(\chi_a, r_\perp)$ provided that we make the approximation $(\chi - \chi_b) \approx (\chi - \chi_a)$. Then

$$-\int_0^{\chi_a} d\chi(\chi_a - \chi)(\chi_b - \chi)[I_J(\chi) - I_0(\chi)]$$
$$\approx \int_0^{\chi_a} d\chi(\chi_a - \chi)^2 \int \frac{dk}{\pi} k^3 P_\Phi(k,\chi)\left[1 - J_0(k\chi\theta)\right]$$
$$= T_2(\chi_a, r_\perp). \qquad (A28)$$

Putting all the pieces together we have then shown that

$$T = T_1(\chi_a, r_\parallel) + T_2(\chi_a, r_\perp). \qquad (A29)$$

Finally to recover the matrix decomposition of the Z^{ij} correlator we can appeal to the simmetry of the lensing displacements ζ. Notice in fact that from Eqs. (A9, A10)

the displacements are clearly invariant with respect to a rotation around the line of sight. We can then arbitrarily pick the coordinate system so that the sources lie in the $x - z$ plane, with the z axis directed along line of sight to $\vec{x}_a$. The coordinates of the sources are then $\vec{x}_a = (0, 0, \chi_a)$ and $\vec{x}_b = (r_\perp, 0, \chi_a + r_\parallel)$ respectively, and since the displacement vector $\vec{r}$ makes an angle α with the line of sight we have that $r_\perp = r \sin(\alpha)$ and $r_\parallel = r \cos(\alpha)$. From here it is then straightforward to obtain the expression for Z^{ij} as in the second line of Eq. (7).

3. Derivation of Eq. (14)

Finally notice that the correction term appearing in Eq. (6) due to lensing can be recast in the form

$$
\begin{aligned}
\langle AB \rangle_2 &\approx \frac{Z^{ij}}{r^2} \left[r_i r_j \frac{d^2 \langle AB \rangle}{dr^2} + r \frac{d\langle AB \rangle}{dr} \left(\delta_{ij} - \frac{r_i r_j}{r^2} \right) \right] \\
&= \frac{Z^{ij}}{r^2} \left[r_i\, r_j\, r \frac{d}{dr} \left(\frac{1}{r} \frac{d\langle AB \rangle}{dr} \right) \right. \\
&\quad \left. + r \frac{d\langle AB \rangle}{dr} \delta^{ij} \right].
\end{aligned}
\tag{A30}
$$

Then, using Eq. (7) for Z^{ij}, which is valid with this choice of coordinates, we get

$$
\begin{aligned}
\langle AB \rangle_2 &= r \frac{d}{dr} \left(\frac{1}{r} \frac{d\langle AB \rangle}{dr} \right) \\
&\quad \times \left[\left(T - \frac{D}{2} \right) \frac{r_\perp^2}{r^2} - (V_a + V_b) \frac{r_\perp r_\parallel}{r^2} + S \frac{r_\parallel^2}{r^2} \right] \\
&\quad + \frac{1}{r} \frac{d\langle AB \rangle}{dr} (2T + S),
\end{aligned}
\tag{A31}
$$

which reduces to Eq. (14) once we notice that T_2 and D provide the dominant contributions.

APPENDIX B: THE EXACT BUT MORE RESTRICTIVE TREATEMENT

Following the approach of [13], it is possible to obtain an "exact" expression for the exponential appearing in Eq. (3) under the restrictive condition that the quantity $k_i (\zeta_a^i - \zeta_b^i)$ is gaussian distributed. If this condition holds then we can use the fact that

$$
\langle e^{iy} \rangle = e^{-\langle y^2 \rangle / 2},
\tag{B1}
$$

to rewrite Eq. (3) as

$$
\begin{aligned}
\langle \tilde{A} \tilde{B} \rangle &= \int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x}_a - \vec{x}_b)} P_{AB}(k, \eta_a, \eta_b) \\
&\quad \times e^{-\frac{1}{2} \langle [\vec{k} \cdot (\vec{\zeta}_a - \vec{\zeta}_b)]^2 \rangle}.
\end{aligned}
\tag{B2}
$$

We therefore need to calculate the value of the correlator

$$
\begin{aligned}
\left\langle \left[\vec{k} \cdot (\vec{\zeta}_a - \vec{\zeta}_b) \right]^2 \right\rangle &= k_i k_j \left[\langle \zeta_a^i \zeta_a^j \rangle + \langle \zeta_b^i \zeta_b^j \rangle - \langle \zeta_a^i \zeta_b^j \rangle \right. \\
&\quad \left. - \langle \zeta_b^i \zeta_a^j \rangle \right].
\end{aligned}
\tag{B3}
$$

It is possible to proceed further by taking into account the fact that the lensing-induced displacements are characterized by different expressions that depend on the direction of the displacement (whether it is parallel or perpendicular to the line of sight). This fact then suggests automatically the adoption of a cylindrical coordinate system for k space. Then, considering for simplicity the case of $k_i k_j \langle \zeta_a^i \zeta_b^j \rangle$ we have

$$
\begin{aligned}
k_i k_j \langle \zeta_a^i \zeta_b^j \rangle &= k_\parallel^2 \langle \zeta_{a,\parallel} \zeta_{b,\parallel} \rangle + k_\parallel k_{i,\perp} \langle \zeta_{a,\parallel} \zeta_{b,\perp}^i \rangle \\
&\quad + k_{j,\perp} k_\parallel \langle \zeta_{a,\perp}^j \zeta_{b,\parallel} \rangle + k_{i,\perp} k_{j,\perp} \langle \zeta_{a,\perp}^i \zeta_{b,\perp}^j \rangle.
\end{aligned}
\tag{B4}
$$

Finally, letting $k_{i,\perp} r_\perp^i = k_\perp r_\perp \cos(\gamma)$ and using Eqs. (A14-A20) it is possible to recast the different terms appearing in the above equation as

$$
k_\parallel^2 \langle \zeta_{a,\parallel} \zeta_{b,\parallel} \rangle = k_\parallel^2 S_{ab},
\tag{B5}
$$

$$
k_\parallel k_{i,\perp} \langle \zeta_{a,\parallel} \zeta_{b,\perp}^i \rangle = -k_\parallel k_\perp \cos(\gamma) V_b,
\tag{B6}
$$

$$
k_{i,\perp} k_{j,\perp} \langle \zeta_{a,\perp}^i \zeta_{b,\perp}^j \rangle = k_\perp^2 \left[T_{ab} - \frac{D_{ab}}{2} \cos(2\gamma) \right].
\tag{B7}
$$

Equation (B2) above then becomes

$$
\begin{aligned}
\langle \tilde{A} \tilde{B} \rangle &= \int \frac{k_\perp dk_\perp dk_\parallel d\gamma}{(2\pi)^3} e^{i[k_\parallel r_\parallel + k_\perp r_\perp \cos(\gamma)]} P_{AB}(k, \eta_a, \eta_b) \\
&\quad \times \exp \left\{ k_\parallel^2 \left(S_{ab} - \frac{S_{aa} + S_{bb}}{2} \right) \right. \\
&\quad + k_\parallel k_\perp \cos(\gamma) (V_a - V_b) \\
&\quad \left. + k_\perp^2 \left[T_{ab} - \frac{D_{ab}}{2} \cos(2\gamma) - \frac{T_{aa} + T_{bb}}{2} \right] \right\},
\end{aligned}
\tag{B8}
$$

where for sake of clarity we have written down explicitly the measure of the integral and we should remind that since the displacement vector $\vec{r}$ makes an angle α with the line of sight $r_\parallel = r \cos(\alpha)$ and $r_\perp = r \sin(\alpha)$.

If we now consider the "purely angular limit" in which only displacements that are perpendicular to the line of sight are taken into account (as in [17]) we restrict ourselves to the case in which the scalar and vector terms of the correlators vanish (the vector term because $V_a = V_b$, the scalar term because $S_{ab} \simeq S_{aa} = S_{bb}$). In this case Eq. (B8) then simplifies considerably into

$$
\begin{aligned}
\langle \tilde{A} \tilde{B} \rangle &= \int \frac{k_\perp dk_\perp d\gamma}{(2\pi)^2} e^{ik_\perp r_\perp \cos(\gamma)} P_{AB}(k, \eta) \\
&\quad \times \exp \left\{ k_\perp^2 \left[T_{ab} - \frac{D_{ab}}{2} \cos(2\gamma) - T_{aa} \right] \right\}
\end{aligned}
\tag{B9}
$$

This expression agrees with the one obtained by Challinor and Lewis [13] for the CMB. The point that need

to be stressed, however, is that it both Eq. (B8) and Eq. (B9) are general expressions that are valid for *any kind of sources*. Finally, the above integral can be carried out exactly by expanding $\exp(D_{ab}\cos(2\gamma)/2)$ in power series and then integrating term by term.

Finally, it seems necessary to point out here the difference between the two approaches and the assumptions that are underlying both of them. The "non-perturbative" approach requires $k_i(\zeta_a^i - \zeta_b^i)$ to be gaussian distributed. Once this price is paid, the apparent reward is to be able to take into account the full exponential, that is all the infinite terms appearing in its series expansion. On the other hand, the "perturbative" approach does not require such an assumption simply because higher order effects would contribute to higher order correlators and these are automatically discarded when the series expansion of the exponential is truncated to second order. It seems necessary to point out, however, that the increased accuracy that can be attained adopting the first approach is actually hard to evaluate. If non-linearities are present (and this is the case when considering that the lensing displacements are proportional to the gradient of the gravitational potential, which goes non-linear at late epochs/low redshift) it is then questionable whether summing all the terms appearing in the exponential would really lead to a *consistently* more accurate result.